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Forces on bodies moving unsteadily in rapidly compressed flows

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The inviscid compressible flow generated by a rigid body of volume $\mathscr{V}$ moving unsteadily with a velocity $\boldsymbol{U}$ in a rapidly compressed homentropic flow is considered. The fluid is compressed isentropically at a rate $\nabla\cdot\boldsymbol{v}_0$ uniformly over a scale much larger than the size of the body and the body moves slowly enough that the Mach number M is low. The flow is initially irrotational and remains so during compression. The perturbation to the flow generated by the body moving unsteadily is non-divergent within an evolving region $\mathscr{D}$ of distance $\int_0^t c_1\,\mathrm{d}t$ from the body, where c_1 is the speed of sound. Within $\mathscr{D}$, the flow is dominated by a source of strength $(\nabla\cdot\boldsymbol{v}_0)\mathscr{V}$ and a dipolar contribution which is independent of the rate of compression, while outside $\mathscr{D}$, compressional waves propagate away from the body. When the body is much smaller than the characteristic distance $\|(\nabla\boldsymbol{v}_0)|_{\boldsymbol{x}_0}\|/\|(\nabla\nabla\boldsymbol{v}_0)|_{\boldsymbol{x}_0}\|$ and the size of the region $\mathscr{D}$, the separation of length scales enables the force on the body to be calculated analytically from the momentum flux far from the body (but within the region $\mathscr{D}$). The contribution to the total force arising from fluid compression is $\rho(t)(\nabla\cdot\boldsymbol{v}_0)\mathscr{V}(\boldsymbol{U}-\boldsymbol{v}_0)\cdot\boldsymbol{\alpha}$, where $\boldsymbol{v}_0$ is the velocity field in the absence of the particles and $\boldsymbol{\alpha}$ is the virtual inertia tensor. Thus a body experiences a drag (thrust) force during fluid compression (expansion) because the density of the fluid displaced forward by the body increases (decreases) with time. The analysis indicates that the sum of the compressional and added-mass force is equal to the rate of decrease of fluid impulse $\boldsymbol{P}=\rho(t)\mathscr{V}(\boldsymbol{U}-\boldsymbol{v}_0)\cdot\boldsymbol{\alpha}$. Thus the concept of fluid impulse naturally extends to the class of flows where the fluid density changes with time, but is spatially uniform.

These new results are applied to consider the inviscid dynamics of a rigid sphere and cylinder projected into a uniformly compressed or expanded fluid. When the fluid rapidly expands, a rigid body ultimately moves with a constant velocity because the total force, which is proportional to the density of the fluid, tends rapidly to zero. When the body moves perpendicular to the axis of compression, it slows down and stops when the density of the fluid is comparable to the density of the body. However, a body moving parallel to the axis of compression is accelerated by pressure gradients which are proportional to fluid density and increases in time.

1. Introduction

In many natural and industrial processess, rigid particles are moving in fluids which are being rapidly compressed or expanded by one or two orders of magnitude.

Normally the effect of compression on particles has been considered in terms of compression waves (e.g. shock and acoustic patches) generated by them when they move fast enough (Chang & Lei 1996). However, there are other processes, such as droplets moving in internal combustion engines, where compression (or expansion) takes place rapidly on a scale much larger than the individual particles. At present there is no theory for these flow problems. The object of this study is to understand the flow generated by the movement of particles and estimate the forces acting on them. Our approach is to extend the previous studies which have looked at the effects of spatial variations of the fluid density (Eames & Hunt 1997; Palierne 1999) to consider the effect of unsteady, but spatially uniform, density fields.

In this paper, we examine the inviscid flow generated by, and force acting on, a rigid body moving in an applied external flow. In the absence of the body, the velocity field $\boldsymbol{v}_0$ may be described locally by a Taylor expansion about a fixed reference point, $\boldsymbol{x}_0$:

$$\boldsymbol{v}_0(\boldsymbol{x},t)=\boldsymbol{v}_0(\boldsymbol{x}_0)+(\boldsymbol{x}-\boldsymbol{x}_0)\cdot(\nabla\boldsymbol{v}_0)|_{\boldsymbol{x}_0}+\tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0):(\nabla\nabla\boldsymbol{v}_0)|_{\boldsymbol{x}_0}+\ldots. \tag{1.1}$$

Within a distance $\mathscr{L}_{\boldsymbol{v}}$ from $\boldsymbol{x}_0$, where $\mathscr{L}_{\boldsymbol{v}}=\|(\nabla\boldsymbol{v}_0)|_{\boldsymbol{x}_0}\|/\|(\nabla\nabla\boldsymbol{v}_0)|_{\boldsymbol{x}_0}\|$ characterizes the distance over which the velocity field varies, Taylor's expansion gives to leading order

$$\boldsymbol{v}_0(\boldsymbol{x},t)=\boldsymbol{v}_0(\boldsymbol{x}_0)+(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{A}, \tag{1.2}$$

where $\boldsymbol{A}=(\nabla\boldsymbol{v}_0)|_{\boldsymbol{x}_0}$. A number of authors (e.g. Taylor 1928*a*; Auton, Hunt & Prud'homme 1988) have examined the forces acting on rigid bodies moving with velocity $\boldsymbol{U}$ in the incompressible inviscid flow (1.2) where $\nabla\cdot\boldsymbol{v}_0=0$. Under the restriction that fluid density ρ is constant and the flow variations over distances comparable to the body lengthscale, a, are small (i.e. $\|\boldsymbol{A}\|a/|\boldsymbol{U}-\boldsymbol{v}_0|\ll 1$), Auton *et al.* (1988) showed that a point symmetric body is subject to the force

$$\boldsymbol{F}_I=\rho(1+C_M)\mathscr{V}\left(\frac{\partial\boldsymbol{v}_0}{\partial t}+(\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_0\right)\bigg|_{\boldsymbol{x}=\boldsymbol{x}_B}-\rho C_M\mathscr{V}\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t}-\rho C_L\mathscr{V}(\boldsymbol{U}-\boldsymbol{v}_0(\boldsymbol{x}_B))\times\boldsymbol{\omega}, \tag{1.3}$$

where the vorticity, $\boldsymbol{\omega}=\nabla\times\boldsymbol{v}_0$, is uniform (according to (1.2)) and $\mathscr{V}$ is the body volume. The added-mass or virtual inertia coefficient C_M characterizes the shape and orientation of the body to the relative flow (Batchelor 1967, p. 407) and assumes the value $C_M=1/2$ for a sphere and $C_M=1$ for a cylinder. The components of the inviscid force (1.3) are the added-mass force

$$\rho C_M\mathscr{V}\left(\frac{\partial\boldsymbol{v}_0}{\partial t}-\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t}\right), \tag{1.4}$$

caused by the relative acceleration of the body to the fluid, the buoyancy force

$$\rho\mathscr{V}\frac{\partial\boldsymbol{v}_0}{\partial t}, \tag{1.5}$$

and the gradient of the ambient pressure

$$\rho(1+C_M)\mathscr{V}(\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_0, \tag{1.6}$$

and lift force caused by the relative motion of the body in a weakly sheared flow

$$-\rho C_L\mathscr{V}(\boldsymbol{U}-\boldsymbol{v}_0)\times\boldsymbol{\omega}, \tag{1.7}$$

(see Magnaudet & Eames 2000). Auton (1987) calculated analytically the lift force on a sphere moving in a weak shear by evaluating the far-field momentum flux and showed that the corresponding lift coefficient is $C_L=1/2$, and confirmed this result numerically. The added-mass, buoyancy and force due to gradients of ambient pressure may be generalized to arbitrarily shaped bodies, however, (1.7) strictly applies only to bodies axisymmetric about $(\boldsymbol{U}-\boldsymbol{v}_0)$. Under this condition, the lift coefficient associated with the shear-induced lift force is C_M. Particularly relevant to the future discussion in this paper about the effect of compression, is the added-mass force associated with an arbitrarily shaped body, which may be expressed more generally as the rate of change of fluid impulse, $\boldsymbol{P} = \rho\mathscr{V}(\boldsymbol{U}-\boldsymbol{v}_0)\cdot\boldsymbol{\alpha}$, through

$$-\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}, \tag{1.8}$$

where the fluid density ρ is constant. Equation (1.8) applies when the volume and shape of the body are functions of time. For point symmetric bodies, $\boldsymbol{\alpha} = C_M\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix and (1.8) reduces to (1.4).

In this paper, we examine the flow generated by a rigid body moving in a fluid undergoing uniform compression or expansion. This problem is rendered tractable by considering a uniformly compressed irrotational flow and by calculating the perturbations to this flow resulting from a body moving unsteadily. A crucial step in the following analysis will be to demonstrate that the perturbation flow is non-divergent in an evolving large region $\mathscr{D}$ enclosing the body within which the flow is determined from Laplace's equation. The problem is stated in its general form in § 2. The flow far from the body (but within the region $\mathscr{D}$) is calculated in § 4 and shown to be dominated by a source term which is proportional to the divergence of the ambient flow and the body volume. The force acting on the body is calculated in § 5, by application of the momentum integral theorem to calculate the far-field momentum flux. The principal result of this paper is that the expression for the added-mass force (1.8) is still applicable when the fluid density changes uniformly with time. These results are described in a general context in § 6.

2. Problem definition

We proceed to examine the compressible inviscid flow $\boldsymbol{v}(\boldsymbol{x}, t)$ around a rigid body moving with a velocity $\boldsymbol{U} = (U_1, U_2, U_3)$. We restrict our analysis to flows where, far from the body, there is uniform compression at a constant rate, and when the principal axes of compression coincide with the Cartesian coordinate axes $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ so that $\boldsymbol{A}$ is a diagonal matrix independent of time,

$$\boldsymbol{A} = \begin{bmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{bmatrix}. \tag{2.1}$$

To calculate the effect of compression on the leading-order flow field and on the force acting on the body, we consider a body which moves impulsively from rest in an irrotational flow field, in order that the irrotationality condition is preserved. A body moving from rest generates compressional waves which propagate away from the body with the speed of sound, c. In the absence of an external compressional flow, the flow is incompressible within an evolving region of distance $O(ct)$ from the body, beyond which is it necessary to account for the compressional waves (see the discussion by Lighthill 1978). When the body moves unsteadily in an irrotational flow characterized by a low Mach number, the radiated compressional waves have a negligible effect on the force acting on the body, which is determined by the acceleration of the body and the local acceleration of the flow. In this paper, we calculate the perturbation flow within an evolving region $\mathscr{D}$ enclosing a body moving unsteadily in a uniformly compressed flow where the perturbation flow is non-divergent, and beyond which compressional waves propagate.

The density and velocity fields satisfy the conservation of mass,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho(\nabla \cdot \boldsymbol{v}), \tag{2.2}$$

and conservation of linear momentum,

$$\rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla p. \tag{2.3}$$

We assume the fluid to be homentropic and the compression adiabatic so that the entropy of the gas is homogeneous and does not change during compression. The equation of state describing adiabatic compression or expansion is

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{2.4}$$

where the subscript 0 refers to an initial state.

The flow satisfies the kinematic condition

$$\boldsymbol{v} \cdot \hat{\boldsymbol{n}} = \boldsymbol{U} \cdot \hat{\boldsymbol{n}}, \tag{2.5}$$

on the body surface S_B where $\hat{\boldsymbol{n}}$ is the unit vector normal to the surface of the body. In the far field, the flow tends to the imposed external flow,

$$\boldsymbol{v} \to \boldsymbol{v}_0 = \boldsymbol{v}_0(\boldsymbol{x}_B) + (\boldsymbol{x} - \boldsymbol{x}_B) \cdot \boldsymbol{A}. \tag{2.6}$$

The centre of volume of the body moves from $\boldsymbol{x}_B(0)$ to $\boldsymbol{x}_B(t) = \boldsymbol{x}_B(0) + \int_0^t \boldsymbol{U}\,\mathrm{d}t$, in time t. We introduce the coordinate $\boldsymbol{x}'$ relative to the body, $\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{x}_B(t)$, so that $\boldsymbol{x}' = \boldsymbol{0}$ corresponds to the body's centre of volume, i.e.

$$\int_{\mathscr{V}} \boldsymbol{x}'\,\mathrm{d}V = \boldsymbol{0}. \tag{2.7}$$

According to (2.4) the fluid is barotropic so that no baroclinic torque acts on the flow and vorticity is generated either by stretching and tilting of fluid elements, or by the compression or expansion of the flow. For the particular case that the external flow is initially irrotational ($\boldsymbol{\omega}(\boldsymbol{x}, 0) = \boldsymbol{0}$), the flow everywhere remains irrotational. As a consequence of irrotationality, the flow in the fluid frame of reference may be expressed in terms of a perturbation velocity $\nabla\phi$ and the undisturbed velocity in this frame, $\boldsymbol{v}_0 = \boldsymbol{x} \cdot \boldsymbol{A}$,

$$\boldsymbol{v} = \boldsymbol{x} \cdot \boldsymbol{A} + \nabla\phi. \tag{2.8}$$

Uniform compression can, in principle, be generated mechanically by squashing fluid between two plane rigid walls. Wang & Kassoy (1990) considered the flow generated by a piston accelerating from rest and observed the generation of compressional waves on the piston surface which travelled rapidly backwards and forwards in the bounded domain. However, the generation of a uniformly compressed flow in the problem we consider would require energy absorbing boundaries to damp compressional waves moving backwards and forwards in a confined space.

3. Density and pressure field

Using the equation of state (2.4), Euler's equation (2.3) in the fluid frame of reference is

$$-\frac{p_0\gamma}{\rho_0(\gamma-1)}\nabla\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} \equiv \frac{\partial \boldsymbol{v}}{\partial t} + \tfrac{1}{2}\nabla v^2 - \boldsymbol{v}\times\boldsymbol{\omega}, \tag{3.1}$$

where $v = |\boldsymbol{v}|$. The flow is irrotational so that the vortex force on the fluid elements is zero and integrating (3.1) gives

$$-\frac{p_0\gamma}{\rho_0(\gamma-1)}\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = B(t) + \left(\frac{\partial\phi}{\partial t} + \tfrac{1}{2}v^2\right), \tag{3.2}$$

where $B(t)$ is an arbitrary constant of integration which we express in terms of a reference density $\rho_1(t)$ as $B(t) = -p_0\gamma(\rho_1/\rho_0)^{\gamma-1}/\rho_0(\gamma-1)$. The speed of sound c is defined as $c^2 \equiv \mathrm{d}p/\mathrm{d}\rho = \gamma p/\rho$. From (3.2), the density field is

$$\rho = \rho_1(t)\left[1 - \frac{\gamma-1}{c_1^2}\left(\frac{\partial\phi}{\partial t} + \tfrac{1}{2}v^2\right)\right]^{1/(\gamma-1)}, \tag{3.3}$$

where the speed of sound corresponding to fluid of density ρ_1 is $c_1 = \sqrt{\gamma p_0 \rho_1^{\gamma-1}/\rho_0^{\gamma}}$. The flow is calculated by expressing conservation of mass as

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} + \boldsymbol{v}\boldsymbol{\cdot}\nabla\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = -(\gamma-1)\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\nabla\boldsymbol{\cdot}\boldsymbol{v}, \tag{3.4}$$

and substituting (3.3) into (3.4):

$$(\gamma-1)(1-\mathscr{C})\left[\frac{1}{\rho_1}\frac{\partial\rho_1}{\partial t} + (\nabla\boldsymbol{\cdot}\boldsymbol{v}_0)\right] = \frac{\mathrm{D}\mathscr{C}}{\mathrm{D}t} - (1-(\gamma-1)\mathscr{C})\nabla^2\phi, \tag{3.5}$$

where $\mathscr{C} = (\phi_t + v^2/2)/c_1^2$.

The Mach number M is defined as the ratio of the characteristic velocity scale to the speed of sound, $M = v/c_1$. According to the flow field (2.8), the velocity increases away from the origin, so that $M = \max\{|\boldsymbol{x}||\nabla\boldsymbol{\cdot}\boldsymbol{v}_0|/c_1, |\boldsymbol{U}|/c_1\}$. We restrict the analysis to the case of low Mach number, $M \ll 1$, necessarily requiring that the flow domain is smaller than $c_1/|\nabla\boldsymbol{\cdot}\boldsymbol{v}_0|$, and that $|\phi_t/c_1^2| \sim |\dot{\boldsymbol{U}}|a/c_1^2 \ll 1$, so that $|\mathscr{C}| \ll 1$. Under these assumptions, (3.3) may be expanded:

$$\rho = \rho_1(t)\left[1 - \frac{1}{c_1^2}\left(\frac{\partial\phi}{\partial t} + \tfrac{1}{2}v^2\right) + O(M^4)\right]. \tag{3.6}$$

According to (3.6), the density field is to leading order uniform and takes the value $\rho_1(t)$, with an $O(M^2\rho_1)$ variation due to pressure gradients generated locally by the body. In this paper we consider a body moving unsteadily in a uniformly compressed fluid. The left-hand side of (3.5) describes the changes in the bulk density of the fluid caused by the uniform compression. When $M \ll 1$ and $|\dot{\boldsymbol{U}}|a/c_1^2 \ll 1$, the conservation of mass reduces to a simpler form which may be integrated

$$\frac{\partial\rho_1}{\partial t} + \rho_1(\nabla\boldsymbol{\cdot}\boldsymbol{v}_0) = 0 \Rightarrow \rho_1 = \rho_0\exp\left(-\nabla\boldsymbol{\cdot}\boldsymbol{v}_0 t\right), \tag{3.7}$$

where the fluid density changes everywhere at the same rate, a quite different type of compressible flow from one with a shock wave and large variations in the rate of compression of fluid elements (Chang & Lei 1996).

Consistent with (3.6), the pressure field is

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma} = p_1(t) - \rho_1(t)\left(\frac{\partial \phi}{\partial t} + \tfrac{1}{2}v^2\right) + O(\rho_1(t)v^2M^2), \tag{3.8}$$

(e.g. see Milne-Thompson 1968, p. 16). The static pressure, $p_1 = p_0(\rho_1/\rho_0)^{\gamma}$, increases everywhere with time because the fluid is being compressed and the dynamic pressure varies spatially due to inertial forces.

For uniform compression, (3.5) reduces to the wave equation,

$$\frac{\partial}{\partial t}\left(\frac{1}{c_1^2(t)}\frac{\partial \phi}{\partial t}\right) - \nabla^2\phi = -\boldsymbol{v}\cdot\nabla\mathscr{C} - (\gamma-1)\mathscr{C}\nabla^2\phi - \frac{\partial}{\partial t}\left(\frac{v^2}{2c_1^2}\right), \tag{3.9}$$

where the nonlinear terms on the right-hand side may be interpreted as a forcing of the flow (see Howe 2003). When $M \ll 1$ and $|\dot{\boldsymbol{U}}|a/c_1^2 \ll 1$, the terms on the right-hand side are negligible compared to those on the left-hand side of (3.9) because $|\mathscr{C}| \ll 1$, and the velocity potential is described by the linear wave equation

$$\frac{\partial}{\partial t}\left(\frac{1}{c_1^2(t)}\frac{\partial \phi}{\partial t}\right) - \nabla^2\phi = 0. \tag{3.10}$$

The effect of compression is implicitly included here through the speed of sound which changes with time. Note that the derivation of the above wave equation requires both the Mach number to be small and the acceleration of the body to be smaller than c_1^2/a – the second constraint is generally ignored (e.g. see Lamb 1932).

When the fluid is not compressed ($\nabla\cdot\boldsymbol{v}_0 = 0$), the wave speed c_1 is constant and

$$\frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = 0. \tag{3.11}$$

Lamb (1932, p. 523) discusses the calculation of the velocity potential for a sphere (of radius a) moving impulsively from rest with a constant speed U and showed that compressional waves are generated which propagate away from the sphere. In the evolving region $\mathscr{D}$, where $|\boldsymbol{x}'| \ll t/c$, moving with the sphere, the flow is steady, incompressible and characterized by a dipole moment $Ua^3/2$. In this region, the component of the velocity potential associated with the compressional wave tends exponentially to zero over a time scale of a/c. The divergence of the flow ($\nabla^2\phi$) is associated with the compressional waves emitted as the sphere accelerates from rest and is only important outside $\mathscr{D}$.

In the problem we consider, the fluid is compressed uniformly. Decomposing the velocity potential in terms of the non-divergent component ϕ_0 and the residual component, ϕ_1, where

$$\phi = \phi_0 + \phi_1, \tag{3.12}$$

the components of the velocity potential satisfy

$$\nabla^2\phi_0 = 0, \qquad \frac{\partial}{\partial t}\left(\frac{1}{c_1^2}\frac{\partial \phi_1}{\partial t}\right) - \nabla^2\phi_1 = \frac{\partial}{\partial t}\left(\frac{1}{c_1^2}\frac{\partial \phi_0}{\partial t}\right). \tag{3.13}$$

When the body starts impulsively from rest, compressional waves are generated locally and propagate away from the body. Following the passage of the initial compressional waves, we see from (3.13) that within an evolving region $\mathscr{D}$ (having a distance much less than $\mathscr{L}_c = \int_0^t c_1 \mathrm{d}t$ from the body), the compressional waves (described by ϕ_1) are generated (or forced) by the local (unsteady) acceleration field of the translating body (corresponding to the right-hand side of (3.13)), so that

ϕ_1 scales as $a^2\mathscr{V}|\ddot{\boldsymbol{U}}|/c_1^2$. When $|\ddot{\boldsymbol{U}}| \ll |\boldsymbol{U} - \boldsymbol{v}(\boldsymbol{x}_B)|c_1^2/a^2$, the component of the flow associated with the compressional waves is negligible compared to the non-divergent component. Implicit in our discussion is that the waves are generated locally by the body and propagate away from the body – the influence of waves generated in the far field and moving towards the sphere is neglected in this discussion. Thus in the limit $M \ll 1$, $a|\dot{\boldsymbol{U}}|/c_1^2 \ll 1$ and $|\ddot{\boldsymbol{U}}| \ll |\boldsymbol{U} - \boldsymbol{v}(\boldsymbol{x}_B)|c_1^2/a^2$, the resulting irrotational flow is non-divergent in the evolving domain $\mathscr{D}$. Although the perturbation flow satisfies Laplace's equation within $\mathscr{D}$, we proceed to consider the limiting case when $\nabla^2\phi = 0$ is satisfied everywhere.

The divergence of the velocity field $\boldsymbol{v}$ is generated by the imposed external flow, so that $\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{v}_0$. The system of equations we solve can now be stated as:

$$\boldsymbol{v} = \boldsymbol{v}_0(\boldsymbol{x}_B) + \boldsymbol{x}'\cdot\boldsymbol{A} + \nabla\phi, \tag{3.14}$$

$$\boldsymbol{v}\cdot\hat{\boldsymbol{n}} = \boldsymbol{U}\cdot\hat{\boldsymbol{n}} \quad \text{on} \quad S_B; \qquad \boldsymbol{v} \to \boldsymbol{v}_0(\boldsymbol{x}_B) + \boldsymbol{x}'\cdot\boldsymbol{A} \quad \text{as} \quad |\boldsymbol{x}'|/a \to \infty, \tag{3.15}$$

$$\nabla^2\phi = 0, \tag{3.16}$$

$$p = p_1(t) - \rho\left(\frac{\partial\phi}{\partial t} + \tfrac{1}{2}v^2\right), \tag{3.17}$$

$$\rho = \rho_0\exp(-\nabla\cdot\boldsymbol{v}_0 t). \tag{3.18}$$

4. Flow around a body moving in a uniformly compressed fluid

We can deduce some general properties of the velocity potential, ϕ, that result from the kinematic condition (3.15). Since the body is impermeable, the volume flux through the body surface S_B is identically zero:

$$0 = \int_{S_B}\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S = \int_{S_B}\nabla\phi\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S + \int_{S_B}(\boldsymbol{x}'+\boldsymbol{x}_B)\cdot\boldsymbol{A}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S = \int_{S_B}\frac{\partial\phi}{\partial n}\,\mathrm{d}S + \mathrm{Trace}(\boldsymbol{A})\mathscr{V}. \tag{4.1}$$

Thus the strength of the source term, Q, required to satisfy the boundary condition on the body surface is

$$Q = \int_{S_B}\frac{\partial\phi}{\partial n}\,\mathrm{d}S = -(\nabla\cdot\boldsymbol{v}_0)\mathscr{V}, \tag{4.2}$$

where $\nabla\cdot\boldsymbol{v}_0 = \mathrm{Trace}(\boldsymbol{A})$; (4.2) is independent of the shape of the body, but is proportional to its volume. Moreover, since there are no sources or sinks in the external flow, the volume flux out of any surface enclosing the body (but within $\mathscr{D}$) is equal to Q.

The kinematic condition (3.15) becomes

$$\nabla\phi\cdot\hat{\boldsymbol{n}} = -(\boldsymbol{x}'\cdot\boldsymbol{A})\cdot\hat{\boldsymbol{n}} + (\boldsymbol{U} - \boldsymbol{v}_0(\boldsymbol{x}_B))\cdot\hat{\boldsymbol{n}}. \tag{4.3}$$

The dipole strength, $\boldsymbol{\mu}$, characterzing the far-field potential flow (Taylor 1928*b*; Batchelor 1967, p. 403) corresponds to that given in the absence of compression

$$\Omega\boldsymbol{\mu} = (\boldsymbol{U} - \boldsymbol{v}_0(\boldsymbol{x}_B))\mathscr{V}\cdot(\boldsymbol{\alpha} + \boldsymbol{I}), \tag{4.4}$$

since it is only determined by the translational velocity of the body. Here Ω takes the value 2π and 4π in two- and three-dimensions respectively, and $\boldsymbol{I}$ is the identity matrix. The integral of ϕ over the surface of the body, $\boldsymbol{I}_B$, is defined by

$$\boldsymbol{I}_B = \int_{S_B}\phi\hat{\boldsymbol{n}}\,\mathrm{d}S,$$

where $\hat{\mathbf{n}}$ is unit vector normal to the body surface and directed into the body (Saffman 1992, p. 74). When the fluid density is constant and takes the value of unity, $\boldsymbol{I}_B$ is identified as the fluid impulse (Batchelor 1967, p. 408), and is given by

$$\boldsymbol{I}_B = (\boldsymbol{U} - \boldsymbol{v}_0(\boldsymbol{x}_B))\cdot\boldsymbol{\alpha}. \tag{4.5}$$

It can be easily shown from (4.4), that the impulse is unchanged by compression since it is only determined by the dipolar component of the far-field. Far from the body but within the region $\mathscr{D}$, the flow is to leading order determined by the source and dipolar contributions to the velocity potential. In two-dimensions, the far-field flow in the frame moving with the body tends to

$$\boldsymbol{v} \to \boldsymbol{v}_0(\boldsymbol{x}_B) + \boldsymbol{x}'\cdot\boldsymbol{A} + \nabla\left(\frac{Q}{2\pi}\log r - \frac{\boldsymbol{\mu}\cdot\boldsymbol{x}'}{r^2}\right), \tag{4.6}$$

and in three-dimensions to

$$\boldsymbol{v} \to \boldsymbol{v}_0(\boldsymbol{x}_B) + \boldsymbol{x}'\cdot\boldsymbol{A} + \nabla\left(-\frac{Q}{4\pi r} - \frac{\boldsymbol{\mu}\cdot\boldsymbol{x}'}{r^3}\right). \tag{4.7}$$

The dipole moment and source strength, $\boldsymbol{\mu}$† and Q, are given by (4.4) and (4.2), respectively. For the case of a cylinder moving parallel to the x-axis with speed U, the velocity potential may be calculated explicitly:

$$\phi(r,\theta) = \underbrace{-(l_1+l_2)\frac{a^2}{2}\log r}_{\text{SOURCE}} \underbrace{-\,U(1-l_1x_B/U)\frac{a^2\cos\theta}{r}}_{\text{DIPOLE}} + \underbrace{\frac{(l_1-l_2)a^3}{4r^2}\cos 2\theta}_{\text{QUADRUPOLE}}, \tag{4.8}$$

where $x' = r\cos\theta$ and $y' = r\sin\theta$. The velocity potential corresponding to flow past a sphere moving parallel to the x-axis in an external flow uniformly compressed along the x' and y' axis is

$$\phi(r,\theta,\varphi) = \underbrace{(l_1+l_2)\frac{a^3}{3r}}_{\text{SOURCE}} \underbrace{-\,U(1-l_1x_B/U)\frac{a^3\cos\theta}{2r^2}}_{\text{DIPOLE}} + \underbrace{\frac{l_2a^4}{18r^3}P_2^2(\cos\theta)\cos 2\varphi - \frac{(l_1+l_2)a^4}{9r^3}P_2^0(\cos\theta)}_{\text{QUADRUPOLE}}, \tag{4.9}$$

where the coordinate axes are $x' = r\cos\theta$, $y' = r\sin\theta\cos\varphi$, and $z' = r\sin\theta\sin\varphi$. The Legendre polynomials are defined as $P_2^2(\cos\theta) = 3\sin^2\theta$, $P_2^0(\cos\theta) = (3\cos^2\theta - 1)/2$ (Arfken 1985).

5. Force on a body moving in a uniformly compressed fluid

The force, $\boldsymbol{F}$, acting on the body moving in an inviscid fluid is caused by the pressure variation over the body surface and is thus expressed as

$$\boldsymbol{F} = \int_{S_B} p\hat{\boldsymbol{n}}\,\mathrm{d}S, \tag{5.1}$$

† A subtle point is that Batchelor (1967, p. 400) defines the doublet strength for two-dimensional flows to have an opposite sign, or sense, to three-dimensional flows. The notation employed here is more consistent, with the sign or sense being the same.

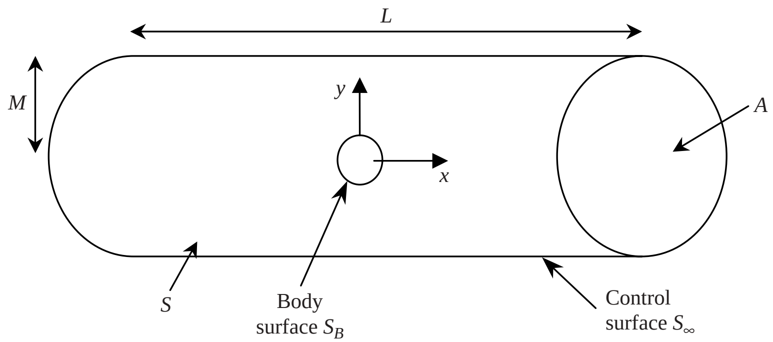

FIGURE 1. Schematic of the control volume used in the calculation of the force acting on the body.

where $\hat{\boldsymbol{n}}$ is the unit normal directed into the body. For the case of a cylinder or sphere moving in a uniformly compressed flow, the force acting on the body may be calculated using expressions for the flow, (4.8), (4.9), to evaluate the pressure distribution over the cylinder or sphere surface. Calculation of the pressure field in inhomogenous flows is generally non-trivial. In such cases, the force may be calculated either from the virtual work done in displacing the body (as described by Taylor 1928*a*) or by calculating the flux of momentum away from the body. Howe (1995) demonstrated how the force on bubbles and bodies moving in inviscid and viscous fluids may be calculated from the momentum flux out of a control volume $\mathscr{V}_\infty$ surrounding the body. The choice of the control volume $\mathscr{V}_\infty$ is arbitrary, but to simplify the evaluation of the far-field momentum flux, we prescribe $\mathscr{V}_\infty$ to be a rectangle (in two dimensions) or a cylinder (in three dimensions), centred on the origin ($\boldsymbol{x} = \boldsymbol{0}$), whose width (or radius) W and length L are taken to be much larger than the size of the body, and the control volume to be slender ($L \gg W$) (see figure 1). The size of the control volume is restricted by the condition $a \ll L \ll \mathscr{L}_c(t)$, so that the compressional waves generated by the body moving from rest have left the control volume. The bounding surface of the control volume consists of the ends of cylinder/rectangle, A, and the curved cylinder surface/rectangles sides, S. The force on the body may be evaluated in terms of surface and volume integrals over a control volume:

$$\boldsymbol{F} = -\int_{A\cup S} p\hat{\boldsymbol{n}}\,\mathrm{d}S - \int_{\mathscr{V}_\infty - \mathscr{V}} \rho(t)\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}\,\mathrm{d}V. \tag{5.2}$$

The force on a body moving unsteadily in an inviscid flow with a uniform density field may be calculated by applying Gauss's theorem to (5.2):

$$\boldsymbol{F} = \underbrace{-\int_{A\cup S}\left(p + \tfrac{1}{2}\rho(t)v^2\right)\hat{\boldsymbol{n}}\,\mathrm{d}S - \int_{\mathscr{V}_\infty-\mathscr{V}}\rho(t)\frac{\partial \boldsymbol{v}}{\partial t}\,\mathrm{d}V - \int_{S_B}\rho(t)(\boldsymbol{v}\cdot\hat{\boldsymbol{n}})\boldsymbol{v}\,\mathrm{d}S}_{\text{(I)}}$$
$$\underbrace{-\int_{A\cup S}\rho(t)\left[(\boldsymbol{v}\cdot\hat{\boldsymbol{n}})\boldsymbol{v} - \tfrac{1}{2}v^2\hat{\boldsymbol{n}}\right]\mathrm{d}S}_{\text{(II)}} + \underbrace{\int_{\mathscr{V}_\infty-\mathscr{V}}\rho(t)\boldsymbol{v}(\nabla\cdot\boldsymbol{v})\,\mathrm{d}V}_{\text{(III)}}. \tag{5.3}$$

We have made use of the fact that the density is uniform, so that $\nabla\rho = \mathbf{0}$. The effect of fluid compression is to introduce term III into the integral form of the momentum equation and this contribution has not been included in previous force descriptions. Term III (in (5.3)) results from the increase in momentum within the control volume caused by the rate of increase of fluid density. This is why a body moving in a compressed flow experiences a drag force.

The force acting on a body moving unsteadily in a uniformly compressed flow is shown in the Appendix to be

$$\boldsymbol{F} = \rho(t)(\nabla\cdot\mathbf{v}_0)\boldsymbol{I}_B - \rho(t)\frac{\mathrm{d}\boldsymbol{I}_B}{\mathrm{d}t} + \rho(t)\mathscr{V}\boldsymbol{U}\cdot\boldsymbol{A} - \rho(t)\Omega\boldsymbol{\mu}\cdot\boldsymbol{A}. \tag{5.4}$$

The first term in the expression corresponds to the new contribution due to uniform compression of the flow and is zero in the absence of compression. The second term corresponds to the added-mass term owing to the acceleration of the local flow (see (1.4)). The third term corresponds to the buoyancy force

$$\rho(t)\mathscr{V}\frac{\partial\boldsymbol{v}_0(\boldsymbol{x}_B)}{\partial t} = \rho(t)\boldsymbol{U}\cdot\boldsymbol{A}\mathscr{V},$$

(see (1.5)) and arises because the fluid velocity seen by the body is changing with time. The fourth term on the right-hand side corresponds to the force acting on the body as it moves through a pressure gradient in the ambient flow (see (1.6)). These calculations demonstrate that compression gives rise to an additional force which is not currently included in multiphase flow models. The compressional force gives rise to an additional drag force when the fluid is compressed and a thrust when the fluid is expanded. The additional drag force experienced arises because the body displaces forward fluid of increasing density during compression. The compressional force may equivalently be interpreted in terms of the change in the pressure distribution over the surface of the body caused by compressional flow. Figure 2 shows how the compressional flow tends to retard the flow near the front of the body but speeds up the fluid at the rear, generating a pressure drop between the front and rear of the body so that the body experiences a drag force. The direction of the force is reversed when the fluid expands uniformly.

The total force may be written in a form combining both the added-mass and compressional force component to give

$$\boldsymbol{F} = -\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} + \rho(t)\mathscr{V}\boldsymbol{U}\cdot\boldsymbol{A} - \rho(t)\Omega\boldsymbol{\mu}\cdot\boldsymbol{A}, \tag{5.5}$$

where $\boldsymbol{P} = \rho(t)\boldsymbol{I}_B$. The analysis developed is valid when the relative motion between the body and the fluid is unsteady; however, when the rate of acceleration of the body is too large, compressional waves are significant near the body and the calculation is invalid.

The force on a rigid cylinder or sphere, characterized by $\boldsymbol{\alpha} = C_M\boldsymbol{I}$, moving unsteadily in a uniformly compressed fluid is

$$\begin{aligned}\boldsymbol{F} &= -\frac{\mathrm{d}}{\mathrm{d}t}[\rho(t)C_M\mathscr{V}(\boldsymbol{U}-\boldsymbol{v}_0(\boldsymbol{x}_B))] + \rho(t)\mathscr{V}\boldsymbol{U}\cdot\boldsymbol{A} - \rho(t)(1+C_M)\mathscr{V}(\boldsymbol{U}-\boldsymbol{v}_0(\boldsymbol{x}_B))\cdot\boldsymbol{A}\\ &= \rho(t)(1+C_M)\mathscr{V}\boldsymbol{v}_0(\boldsymbol{x}_B)\cdot\boldsymbol{A} - \rho(t)C_M\mathscr{V}\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} - C_M\mathscr{V}(\boldsymbol{U}-\boldsymbol{v}_0(\boldsymbol{x}_B))\frac{\mathrm{d}\rho}{\mathrm{d}t}.\end{aligned} \tag{5.6}$$

In the absence of compression, (5.6) reduces to the force expression obtained by substituting (2.1) into (1.3). The new effect of fluid compression is described by the last term on the right-hand side of (5.6).

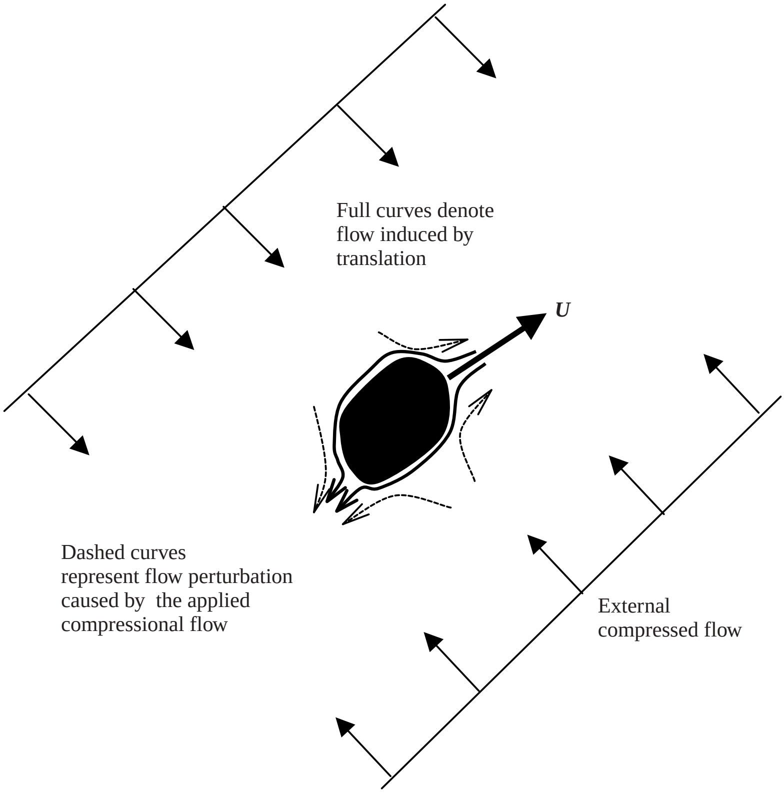

FIGURE 2. Schematic illustrating how a compressional flow leads to a decrease in pressure at the rear of the body and an increase at the front of the body. The consequence is a drag force acting on the body.

Equation (5.6) is now applied to study the effect of compression on the trajectory of a rigid cylinder or sphere projected from the origin with initial velocity $(U_x(0), U_y(0))$ in a flow $\boldsymbol{v}_0(x, y) = (l_1x, l_2y)$. According to (3.7), the density of the fluid is $\rho = \rho_0 \exp(-(l_1 + l_2)t)$. From (5.6), the dynamics of a point symmetric body moving with velocity $\boldsymbol{U} = (U_x, U_y)$ are described by:

$$(\rho_p + C_M\rho(t))\frac{\mathrm{d}U_x}{\mathrm{d}t} = \rho(t)[C_M(l_1 + l_2)U_x + xl_1[l_1 - C_Ml_2]], \tag{5.7}$$

$$(\rho_p + C_M\rho(t))\frac{\mathrm{d}U_y}{\mathrm{d}t} = \rho(t)[C_M(l_1 + l_2)U_y + yl_2[l_2 - C_Ml_1]]. \tag{5.8}$$

The above equations are uncoupled, linear and independent of the volume of the body. Therefore it is only necessary to consider motion parallel to the x-axis to understand the effect a compressional flow on the dynamics of the body. A body moving in an incompressible fluid ($l_1 + l_2 = 0$) accelerates away from the origin towards the region of low pressure and moves a distance x from the origin in time t where $l_1x/U_x(0) = ((\rho_p + C_M\rho)/\rho(1 + C_M))^{1/2} \sinh((\rho(1 + C_M)/(\rho_p + C_M\rho))^{1/2}l_1t)$. When the fluid is compressed along the y-axis ($l_1 = 0$), the body moves a distance x from the origin in time t, where

$$-\frac{l_2x}{U_x(0)} = \left(1 + \frac{C_M\rho_0}{\rho_p}\right)\log\left(\frac{\exp(-l_2t)(1 + C_M\rho_0/\rho_p)}{1 + (C_M\rho_0/\rho_p)\exp(-l_2t)}\right). \tag{5.9}$$

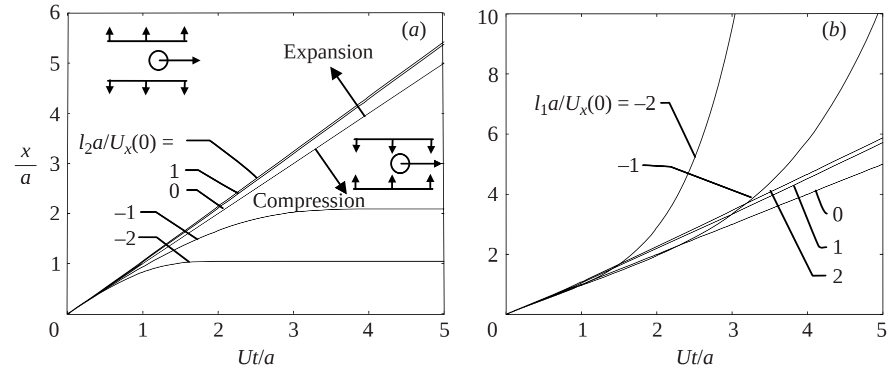

FIGURE 3. Trajectories of a rigid cylinder ($C_m = 1$) of radius a projected into a compressed fluid. The cylinder is projected (*a*) perpendicular and (*b*) parallel to the axis of compression. The ratio of density of the cylinder to the initial density of the fluid is $\rho_p/\rho_0 = 10$.

Thus, when the fluid expands ($l_2 > 0$), the density of the ambient fluid decreases rapidly and the body ultimately moves with a constant speed $(1 + C_M\rho_0/\rho_p)U_x(0)$. However, when the fluid is compressed ($l_2 < 0$), the fluid density increases and the body slows down owing to the compressional force and ultimately stops a distance $-(U_x(0)/l_2)(1+ C_M\rho_0/\rho_p)\log(1 + \rho_p/C_M\rho_0)$ from the origin over a time scale $O(-\log(\rho_p/C_M\rho_0)/l_2)$. Thus, the body ultimately stops when the density of the fluid is comparable to the density of the body. Figure 3(*a*) shows how the trajectory of a rigid cylinder depends on the ratio of the advective to compressional timescale, $l_2a/U_x(0)$, where a is the radius of the cylinder. A rigid sphere projected with the same initial speed as a cylinder slows down over a longer time scale owing to the smaller added-mass coefficient and stops a further distance from the origin.

When the body moves parallel to the compressional axis ($l_2 = 0$), the dynamics are more complex because there is a combination of acceleration by gradients of fluid pressure and the compressional force. When the fluid expands ($l_1 > 0$), the force acting on the body tends exponentially to zero because the density of the fluid decreases, and the body ultimately moves with a constant speed as indicated in figure 3(*b*). When the fluid is compressed, the force generated by gradients of the ambient fluid pressure ultimately dominates the compressional force and the body accelerates away from the origin. In the more general case when the fluid is compressed or expanded about the x- and y-axis, the body ultimately accelerates (or decelerates) away from the origin when $l_1(l_1 - C_Ml_2)$ is positive (or negative).

6. Discussion

In this paper, we have examined the flow generated by a rigid body moving unsteadily in a rapidly compressed flow. To render the analysis tractable, we considered a body starting impulsively from rest in an initially irrotational flow and calculated the flow generated within an evolving region $\mathscr{D}$ (which is much larger than the size of the body) where the flow perturbation is non-divergent. The compression was assumed to be uniform and constant, and the Mach number of the flow is sufficiently low that flow adjustment occurs over a time much shorter than the advective time scale by the generation of compressional waves. The separation of length scales between the size of the body, the characteristic distance over which the external flow field varies ($\mathscr{L}_v$) and the size of the region ($\mathscr{L}_c$), enabled the force acting on the body to be calculated from the momentum flux far from the body, but within the region $\mathscr{D}$. The force is dominated by the monopolar flow field generated by the kinematic condition imposed by the surface of the body, and the dipolar component generated by translation. The impact of compressional waves, generated by the initial rapid acceleration of the body from rest, on the force acting on a moving body are complex. These waves must be considered when calculating the unsteady flow outside $\mathscr{D}$; however, they have a negligible effect on the flow within $\mathscr{D}$ and on the total force acting on the body.

We have shown that bodies moving in rapidly compressed flows experience a drag force as a consequence of displacing forward fluid of increasing density, and this force is not captured by current multiphase models. We have demonstrate that the compressional and added-mass forces may be combined and expressed more generally in terms of the rate of decrease of fluid impulse, $\boldsymbol{P} = \rho(t)\boldsymbol{I}_B$. Although discussed by Saffman (1972), who suggested that the concept of fluid impulse can be applied even when the gradient of fluid density is weak, here we have shown that the fluid impulse concept extends naturally to uniformly compressed fluids, where the density gradient is zero, but the density field changes with time. The additional compressional force may also be deduced by a straightforward application of a Hamiltonian description of the forces on bodies moving in a potential flow (e.g. Palierne 1999) because the kinetic energy of the fluid is contained in the irrotational component of the flow, and the potential energy of the fluid due to compression does not affect the force. A limitation of a Hamiltonian description of this problem is that it provides no insight into the flow.

The calculations presented in this paper relate to bodies moving in initially irrotational compressed flows described by (2.1). The effect of a rotational component to the flow was not considered in this paper. However, when the flow initially has a rotational component, compression tends to increase the strength of the vorticity field because the circulation associated with a closed material circuit is conserved while its area decreases. When the flow is two-dimensional and the vorticity field ω is initially uniform, the vorticity changes uniformly during compression and expansion because ω/ρ is materially conserved. Thus, the flow generated by a cylinder moving in a uniformly compressed fluid corresponds to that calculated by Batchelor (1967, p. 542) and the lift coefficient is unchanged. However, when the flow is three-dimensional, the vorticity field is affected both by advection and stretching by the applied compressional flow and the flow past the body. Auton's expression for lift applies in the limiting case where changes in vorticity due to compression in a time $a/|\boldsymbol{U}-\boldsymbol{v}_0(\boldsymbol{x}_B)|$ are small, that is, $|\nabla\cdot\boldsymbol{v}_0|a/|\boldsymbol{U}-\boldsymbol{v}_0(\boldsymbol{x}_B)| \ll 1$. When this condition is not satisfied, the effect of vorticity generation through compression and advection by the compressional flow must also be considered.

The new inviscid analysis presented in this paper has provided insight into the effect of changes of the ambient fluid density on the dynamics of a rigid body and could be useful for calculating forces on particles in practical problems where it is usual to add empirically derived drag forces to obtain estimates of the total force (e.g. Hunt, Perkins & Fung 1994; Magnaudet & Eames 2000).

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Appendix

Term III

The rate of compression, $\nabla\cdot\boldsymbol{v}_0 = \mathrm{Trace}(\boldsymbol{A})$, is constant. Term III is

$$\rho(t)\mathrm{Trace}(\boldsymbol{A})\int_{\mathscr{V}_\infty-\mathscr{V}}(\boldsymbol{x}\cdot\boldsymbol{A}+\nabla\phi)\,\mathrm{d}V = \rho(t)\mathrm{Trace}(\boldsymbol{A})\left[\int_{\mathscr{V}_\infty-\mathscr{V}}\boldsymbol{x}\cdot\boldsymbol{A}\,\mathrm{d}V\right.$$

$$\left.+\int_{A\cup S}\phi\hat{\boldsymbol{n}}\,\mathrm{d}S+\int_{S_B}\phi\hat{\boldsymbol{n}}\,\mathrm{d}S\right] = \rho(t)\mathrm{Trace}(\boldsymbol{A})\left[-\boldsymbol{x}_B\cdot\boldsymbol{A}\mathscr{V}+\int_{A\cup S}\phi\hat{\boldsymbol{n}}\,\mathrm{d}S+\boldsymbol{I}_B\right].$$

Here, we have made use of the fact that the control volume $\mathscr{V}_\infty$ is symmetric about the origin ($\boldsymbol{x}=0$).

Term II

Using $\boldsymbol{v}=\boldsymbol{x}\cdot\boldsymbol{A}+\nabla\phi$, and expanding the terms in the integrand, we have

$$-\int_{A\cup S}\left((\boldsymbol{v}\cdot\hat{\boldsymbol{n}})\boldsymbol{v}-\tfrac{1}{2}v^2\hat{\boldsymbol{n}}\right)\mathrm{d}S$$

$$=-\int_{A\cup S}[(\boldsymbol{x}\cdot\boldsymbol{A})\cdot\hat{\boldsymbol{n}}]\nabla\phi\,\mathrm{d}S-\int_{A\cup S}(\nabla\phi\cdot\hat{\boldsymbol{n}})(\boldsymbol{x}\cdot\boldsymbol{A})\,\mathrm{d}S+\int_{A\cup S}[(\boldsymbol{x}\cdot\boldsymbol{A})\cdot\nabla\phi]\hat{\boldsymbol{n}}\,\mathrm{d}S. \quad (\mathrm{A}\,1)$$

The other contributions are zero, owing to symmetry of the control volume, or because they decay sufficiently rapidly in the far field. Writing $\boldsymbol{x}=\boldsymbol{x}'+\boldsymbol{x}_B$, we find that (A 1) is

$$-\int_{A\cup S}[(\boldsymbol{x}'\cdot\boldsymbol{A})\cdot\hat{\boldsymbol{n}}\nabla\phi]\,\mathrm{d}S-\int_{A\cup S}(\nabla\phi\cdot\hat{\boldsymbol{n}})(\boldsymbol{x}'\cdot\boldsymbol{A})\,\mathrm{d}S+\int_{A\cup S}[(\boldsymbol{x}'\cdot\boldsymbol{A})\cdot\nabla\phi]\hat{\boldsymbol{n}}\,\mathrm{d}S.$$

$$-\int_{A\cup S}[(\boldsymbol{x}_B\cdot\boldsymbol{A})\cdot\hat{\boldsymbol{n}}\nabla\phi]\,\mathrm{d}S-\int_{A\cup S}(\nabla\phi\cdot\hat{\boldsymbol{n}})(\boldsymbol{x}_B\cdot\boldsymbol{A})\,\mathrm{d}S+\int_{A\cup S}[(\boldsymbol{x}_B\cdot\boldsymbol{A})\cdot\nabla\phi]\hat{\boldsymbol{n}}\,\mathrm{d}S. \quad (\mathrm{A}\,2)$$

Now,

$$\left[\int_{A\cup S}(\nabla\phi\cdot\hat{\boldsymbol{n}})\boldsymbol{x}'\,\mathrm{d}S\right]\cdot\boldsymbol{A}=\left[\int_{A\cup S}[\boldsymbol{x}'(\nabla\phi\cdot\hat{\boldsymbol{n}})-\phi\hat{\boldsymbol{n}}]\,\mathrm{d}S+\int_{A\cup S}\phi\hat{\boldsymbol{n}}\,\mathrm{d}S\right]\cdot\boldsymbol{A}$$

$$=\left[\Omega\boldsymbol{\mu}+\int_{A\cup S}\phi\hat{\boldsymbol{n}}\,\mathrm{d}S\right]\cdot\boldsymbol{A},$$

(from Batchelor 1967, equation (6.4.32), p. 399, and the last equation on p. 400). The third and fourth terms in (A 2) may be evaluated in three dimensions by writing the velocity potential, $\phi=-Q/4\pi r-\boldsymbol{\mu}\cdot\boldsymbol{x}'/r^3$, as a sum of source and dipolar contributions and integrating over the control surface $A\cup S$. Evaluating the fourth and last term in (A 2) shows that their contribution is negligible compared to the other terms. The fifth term in (A 2) is $-(\int_{A\cup S}\nabla\phi\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S)(\boldsymbol{x}_B\cdot\boldsymbol{A})=\mathscr{V}\mathrm{Trace}(\boldsymbol{A})(\boldsymbol{x}_B\cdot\boldsymbol{A})$.

Term II is

$$-\rho(t)\Omega\mu\cdot\boldsymbol{A}-\rho(t)\left(\int_{A\cup S}\phi\hat{\boldsymbol{n}}\,\mathrm{d}S\right)\cdot\boldsymbol{A}-\rho(t)\int_{A\cup S}[((\boldsymbol{x}'\cdot\boldsymbol{A})\cdot\hat{\boldsymbol{n}})\nabla\phi$$

$$-(\boldsymbol{x}'\cdot\boldsymbol{A})\cdot\nabla\phi\hat{\boldsymbol{n}}]\,\mathrm{d}S+\rho(t)\mathscr{V}\mathrm{Trace}(\boldsymbol{A})(\boldsymbol{x}_B\cdot\boldsymbol{A}).$$

The third term in the above expression is evaluated explicitly to give

$$\rho(t)\begin{bmatrix}0\\2\pi\mu_2(l_1+l_3)\\2\pi\mu_3(l_1+l_2)\end{bmatrix}.$$

Term II reduces to

$$-\rho(t)\Omega\mu\cdot\boldsymbol{A}-\rho(t)\left(\int_{A\cup S}\phi\hat{\boldsymbol{n}}\,\mathrm{d}S\right)\cdot\boldsymbol{A}+\rho(t)\begin{bmatrix}0\\2\pi\mu_2(l_1+l_3)\\2\pi\mu_3(l_1+l_2)\end{bmatrix}+\rho(t)\mathscr{V}\mathrm{Trace}(\boldsymbol{A})(\boldsymbol{x}_B\cdot\boldsymbol{A}).$$

Term I

$$-\rho(t)\int_{S_B}\frac{\partial\phi}{\partial t}\hat{\boldsymbol{n}}\,\mathrm{d}S-\rho(t)\int_{S_B}(\boldsymbol{U}\cdot\hat{\boldsymbol{n}})\nabla\phi\,\mathrm{d}S-\rho(t)\int_{S_B}(\boldsymbol{U}\cdot\hat{\boldsymbol{n}})(\boldsymbol{x}\cdot\boldsymbol{A})\,\mathrm{d}S$$

since $\boldsymbol{v}\cdot\hat{\boldsymbol{n}}=\boldsymbol{U}\cdot\hat{\boldsymbol{n}}$ on S_B. In addition,

$$\int_{S_B}(\boldsymbol{U}\cdot\hat{\boldsymbol{n}})(\boldsymbol{x}\cdot\boldsymbol{A})\,\mathrm{d}S=\int_{S_B}(\boldsymbol{U}\cdot\hat{\boldsymbol{n}})[(\boldsymbol{x}_B+\boldsymbol{x}')\cdot\boldsymbol{A}]\,\mathrm{d}S=-\boldsymbol{U}\cdot\boldsymbol{A}\mathscr{V}.$$

Thus, term I is equal to

$$-\rho(t)\frac{\mathrm{d}\boldsymbol{I}_B}{\mathrm{d}t}+\rho(t)\boldsymbol{U}\cdot\boldsymbol{A}\mathscr{V},$$

since $\mathrm{d}\boldsymbol{I}_B/\mathrm{d}t=\int_{S_B}\partial\phi/\partial t\hat{\boldsymbol{n}}\,\mathrm{d}S+\int_{S_B}(\boldsymbol{U}\cdot\hat{\boldsymbol{n}})\nabla\phi\,\mathrm{d}S$.

Total force

The total force is calculated by adding together terms I, II and III:

$$\boldsymbol{F}=-\rho(t)\frac{\mathrm{d}\boldsymbol{I}_B}{\mathrm{d}t}+\rho(t)\boldsymbol{U}\cdot\boldsymbol{A}\mathscr{V}-\rho(t)\Omega\boldsymbol{\mu}\cdot\boldsymbol{A}+\rho(t)\mathrm{Trace}(\boldsymbol{A})\boldsymbol{I}_B$$
$$+\rho(t)\left(\int_{A\cup S}\phi\hat{\boldsymbol{n}}\,\mathrm{d}S\right)\cdot(\mathrm{Trace}(\boldsymbol{A})\boldsymbol{I}-\boldsymbol{A})+\rho(t)\begin{bmatrix}0\\2\pi\mu_2(l_1+l_3)\\2\pi\mu_3(l_1+l_2)\end{bmatrix}.$$

For the cylindrical control volume used,

$$\int_{A\cup S}\phi\hat{\mathbf{n}}\,\mathrm{d}S=\begin{bmatrix}0\\-2\pi\mu_2\\-2\pi\mu_3\end{bmatrix}.$$

The total force is therefore

$$\boldsymbol{F}=\rho(t)(\nabla\cdot\boldsymbol{v}_0)\boldsymbol{I}_B-\rho(t)\frac{\mathrm{d}\boldsymbol{I}_B}{\mathrm{d}t}+\rho(t)\boldsymbol{U}\cdot\boldsymbol{A}\mathscr{V}-\rho(t)\Omega\boldsymbol{\mu}\cdot\boldsymbol{A}.$$

These calculations are confirmed for two-dimensional flows by repeating the above calculations.

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